Report on "American Option Pricing and Hedging Strategies"

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Content

This paper mainly discusses the American option's hedging strategies via binomial model and the basic idea of pricing and hedging American option. Although the essential scheme of hedging is almost the same as European option, small differences may arise when simulating the process for American option holder has more rights, spelling that the option can be exercised at anytime before its maturity. Our method is dynamic-hedging method.

Keywords: put; American put; call; hedging strategies; price; option; binomial model; Black-Scholes model.

I. BRIEF INTRODUCTION

The framework of the paper is as follows. In the first section I'll present a very simple example on how to price American option and how the hedge can be done at length, which is modified from[3]. Then the computer simulation would be represented in section II, following the "real possible path" (if the assumption that the stock price will be two possible value in binomial model holds), say, the underlying simulating stock price was based on binomial model. In section III some useful discussion some comments on pricing and hedging process will be presented.

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II. A SIMPLE EXAMPLE OF HEDGING AMERICAN CALL

To see the whole hedging process distinctly, I give the following simple numerical example, which has adapted from Cox, Rubinstein, Ross [3]. The example is about the American call hedging process, however, the process w.r.t. put is completely the same as this one, except selling short instead of borrowing money to buy stocks. Thus this naive example could reflect the basic idea. In fact the hedging process is a replication of the pricing process, here we use delta hedging maneuver, see reference [1][2][3][4]. If the current market price M ever differed from its formula (theoretic) value C. If $M > C$, we would hedge, and if $M < C$, "reverse hedge", to try to keep profit. Suppose the values of the underlying variables (dollars) are

$S = 100,\ n = 3,\ K = 100,\ a = -0.5,\ b = 1.5,\ r = 0.1,\ M = 45$

In this case, $p = (r - a)/(b - a) = 0.6$. The relevant values of the discount factor are

$r^{-1} = 0.909,\ r^{-2} = 0.826,\ r^{-3} = 0.751$

The paths the stock price may follow and their corresponding probabilities (using probability p) are, when $n = 3$, with $S = 100$,

$$
\begin{array}{ccccccc}
100 & \longrightarrow & 150(.6) & \longrightarrow & 225(.36) & \longrightarrow & 337.5(.216) \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \\
50(.4) & \longrightarrow & 75(.48) & \longrightarrow & 112.5(.432) & & \\
\big\downarrow & & \big\downarrow & & & & \\
25(.16) & \longrightarrow & 37.5(.288) & & & & \\
\big\downarrow & & & & & & \\
12.5(.064) & & & & & &
\end{array}
$$

when $n = 2$, if $S = 150$,

$$
\begin{array}{ccccc}
150 & \longrightarrow & 225(.6) & \longrightarrow & 337.5(.36) \\
\downarrow & & \downarrow & & \\
75(.4) & \longrightarrow & 112.5(.48) & & \\
\downarrow & & & & \\
37.5(.16) & & & &
\end{array}
$$

when $n = 2$, if $S = 50$,

$$
\begin{array}{ccccc}
50 & \longrightarrow & 75(.6) & \longrightarrow & 112.5(.36) \\
\downarrow & & \downarrow & & \\
25(.4) & \longrightarrow & 37.5(.48) & & \\
\downarrow & & & & \\
12.5(.16) & & & &
\end{array}
$$

A. Pricing the option

The pricing process of concrete model depends on the following fact[1]: Suppose the market is viable and complete, the price of American option satisfies nether formula, $C_{n-1} = max(Z_{n-1}, (1+r)^{-1}E^*(C_n|\ \mathcal{P}_{n-1}))$, where C_{n-1} denotes option value at step (time)n, Z_{n-1} denotes profit option holder will get when exercising the option or not, say, $Z_n = max(S_n - K, 0)$(for a call) or $Z_n = max(K - S_n, 0)$(for a put), K strike price, S_n stock price at step n, $\mathcal{P}_{n-1}$ denotes the all the information before n , E^* denotes expectation under some probability measure (risk neutral probability measure). Then computable formula, which can be calculated on computer, follows directly from the above results: Let $C_n = P(n, S_n)$, then $P(N, x) = Z_n$, N is the maturity.

$$P(n, x) = max(Z_n, (1 + r)^{-1}f(n + 1, x))$$

with $f(n+1, x) = pP(n+1, x(1+a)) + (1-p)P(n+1, x(1+b))$ and $p = (b-r)/(b-a)$. From back forward methods, option value at each time can be obtained without any difficulties. Using the matlab program based on the above idea. The current

value of the call would be

$$C = 42.5995.$$

B. Hedging the option

A riskless hedge maneuver can be simply interpreted as follows: Suppose current stock price is S. We form a portfolio containing δ shares of stock and the riskless dollar amount B. This will cost $\delta S + B$. At the end of the period, the value C of this portfolio will be

$P(C = \delta(1+a)S + rB) = p$ denote this value C by C_a ;

$P(C = \delta(1+b)S + rB) = 1 - p$ denote this value C by C_b .

Since we can select δ and B in any way we wish, suppose we choose them to equate the end-of-period values of the portfolio and the call for each possible outcome. This requires that

$$\delta(1+a)S + rB = C_a$$

$$\delta(1+b)S + rB = C_b$$

Solving these equations, we have

$$\delta = \frac{C_b - C_a}{(b-a)S}, B = \frac{(1+a)C_b - (1+b)C_a}{(a-b)S}$$

Using the above formula of δ and matlab program, we have the following tree diagram giving the paths the call value may follow and the corresponding values of δ :

$$\begin{tikzcd}
42.5995(.719) & 75.6198(.848) \arrow[l] & 134.0909(1.00) \arrow[l] & 273.5 \arrow[l] \\
3.7190(.136) \arrow[u] & 6.8182(.167) \arrow[l] \arrow[u] & 12.5 \arrow[l] \arrow[u] & \\
0(.00) \arrow[u] & 0 \arrow[l] \arrow[u] & & \\
0 \arrow[u] & & &
\end{tikzcd}$$

With this preliminary analysis, we are prepared to use the formula to take advantage of mispricing in the market. Suppose that when $n = 3$, the market price of the call is 36. Our formula tells us the call should be worth 42.5995. The option is overpriced, so we could plan to sell it and assure ourselves of a profit equal to the mispricing difference. Here are the steps we could take for a typical path the stock might follow.

Step 1 ($n = 3$): Sell the call for 36. Take 34.065 of this and invest it in a portfolio containing $\delta = 0.719$ shares of stock by borrowing $0.719(100) - 42.5995 = 29.3005$. Take the remainder, $45 - 42.5995 = 2.4005$, and put it in the bank.

Step 2 ($n = 2$): Suppose the stock goes to 150 so that δ the new is 0.848. Buy $0.848 - 0.719 = 0.129$ more shares of stock at 150 per share for a total expenditure of 19.350. Borrow to pay the bill. With an interest rate of 0.1, we already owe $29.3005(1.1) = 32.2306$. Thus, our total current indebtedness is $32.2306 + 19.350 = 51.5806$.

Step 3 ($n = 1$): Suppose the stock price now goes to 75. The δ new is 0.167. Sell $0.848 - 0.167 = 0.681$ shares at 75 per share, taking in $0.681(75) = 51.0750$. Use this to pay back part of our borrowing. Since we now owe $51.5806(1.1) = 56.7387$, the repayment will reduce this to $56.7387 - 51.0750 = 5.6637$.

Step 4.1 ($n = 0$): Suppose the stock price now goes to 37.5. The call we sold has expired worthless. We own 0.167 shares of stock selling at 37.5 per share, for a total value of $0.167(37.5) = 6.2625$. Sell the stock and repay the $5.6637(1.1) = 6.2301$ that we now owe on the borrowing without considering the computing error $(6.2625 - 6.2301 = 0.0324)$. (In fact, such a error can be eliminated with high precision, however in the simulation, error can be cumulated to a little large if the step N goes to infinity while the precision of the computer is fixed. Fortunately, it has limit relating to the precision of the computer, this can be seen in Section II from the hedging performance table). Go back to the bank and take our initial deposit, which has now grown to $2.4005(1.1)^3 = 3.1951$.

Step 4.2 ($n = 0$): Suppose, instead, the stock price goes to 112.5. The call we sold is in the money at the expiration date. Buy back the call, or buy one share of stock and let it be exercised, incurring a loss of $112.5 - 100 = 12.5$ either way. Borrow

to cover this, bringing our current indebtedness to $6.2301 + 12.5 = 18.7301$. We own 0.167 shares of stock selling at 90 per share, for a total value of $0.167(112.50) = 18.7875$. Sell the stock and repay the borrowing without considering the computing error. Go back to the bank and take our initial deposit, which has now grown to $2.4005(1.1)^3 = 3.1951$.

C. Remark:

1. In the above hedging process, we don't care about the trends of stock price whether it goes up or down. Of course, if the stock comes into line we can do best thing for us without any loss.

2. If at any step the real price of option equals its theoretic value, we can buy the option back without concerning of keeping the portfolio adjusted.

3. In conducting option, we assume every man is rational (which is an essential assumption of our simulation.). If the man behaves foolishly and exercises the option at a wrong time, no mater when he/she exercises the option, (for instance, exercising it as soon as possible or until the expiration without carrying out it at optimal time), the value of our portfolio by the above hedging way would always no less than $S-K$, and our simulation will illustrate this in the following section. see [1][3].

4. Instead, we could have made the adjustments by keeping the number of shares of stock constant and buying or selling calls and bonds. However, this could be dangerous since the real option price maybe more than its theoretic value, which next remark mentions. In large, *"To adjust a hedged position, never buy an overpriced option or sell an underpriced option."*[3]

5. The foregoing method is called dynamic hedging strategies(another name "hedge-and-forget scheme") while there exists static hedging strategies(also called "stop-loss strategies"). Simulation proves dynamic scheme is better than static one when applied to European call. Then we can also intuitively expect the same conclusion when employed in American option since its special case is European call. The reason why previous conclusion holds still stands when applied to American option. See [2][4]

III. SIMULATION OF HEDGING OPTION PROCESS

A. Assumption

1. Primary assumption of binomial model.

2. The option holder behaves rationally, or equivalently, the option would be exercised as soon as the optimal time comes (for a call $\delta = 1$, or $S - K \geq$Call Price; for a put $\delta = -1$, or $K - S \geq$Put Price).

3. There're no transactions costs.

4. Short is permitted and the rate of money borrowed is riskless rate.

B. Simulation

Suppose the underlying variables are as follows (the following data is selected from [2][4], they utilize Black-Scholes model to price and hedge European calls, different from our way based on binomial model.):
Current stock price $S_0 = \$49$,
Strike price $K = \$50$,
Stock volatility $\sigma = 20\%$ per annum
Riskless instantaneous interest rate $R = 5\%$per annum
Option time to maturity $M = 20$ weeks (0.3836year)
Stock expected return $\mu = 13\%$ per annum
The real market option price is 3.
N denotes the steps of binomial model (left to be determined);
As we know, if we set

$$r = RT/N, 1 + a = exp(-\sigma\sqrt{T/N}), 1 + b = exp(\sigma\sqrt{T/N}), p = (b - r)/(b - a).$$

The price of option will converge to the solution from Black-Scholes model as N goes to infinity.

C. Simulation Plot:

The **yellow** point denotes the changes of stock price at every step(SP).

TABLE I: Performance of our dynamic hedging (The performance measure is the ratio of the variance of the cost of writing the option and hedging it to the theoretical price of the option)

N steps	5	10	25	50	100	250	500
step length(weeks)	4	2	0.8	0.4	0.2	0.008	0.004
Hedging performance	0.0042	0.0036	0.006	0.0031	0.002	0.001	0.0007
average gain(per share)	0.5571	0.6094	0.6719	0.638	0.6433	0.6338	0.631

The **green** line denotes the total cashes you have or owed without considering the stock value you holds at every step (TC).

The **blue** line denotes the total value of your portfolio containing stock and bonds at every step (TV).

The **red** line denotes the total money you'll receive when the option holder exercises the option or loses the option at every step before optimal time (TM).

For each N, there're four plots including two call-hedge plots and two put-hedge plots respectively. (all the values in the plots are per share), see **APPENDIX**.

From the charts, we could see that the exercising time happens at maturity or before it, and hedging scheme works very well.

D. Hedging performance:

If we fixed $N = 500$, reiterate the simulation 1000 times (see program "gain.m"). The average gain per share of all the gains of each simulation is 0.6307, variance is 0.00063542. Therefore we know the performance of our hedge can be measured by this variance, see [2]. Performance of dynamic hedging strategies is shown in the table I. From the table, we can see such a hedging scheme performs very well for the critical ratio goes to zeros as N goes to infinity.

IV. DISCUSSION AND REMARKS

Remark1. Our simulation shows the dynamic hedging strategies work perfectly if we accept the assumptions. Hedging performance reveals its advantages in hedging American call or put. The program can immediately be applied to simulate the hedging strategies of European call or put with a small modification.

Remark2. In our simulation we have assumed the volatility is constant with time. In fact in short term, this assumption can be seen proper and reasonable, however, in a long term (e.g. hedge a call or put in a long period such as more than one year), errors would probably raised highly, has $sigma$ been constant over the periods. In practice, volatility can be calculated, using the past information of stock price and itself, with statistical methods. Considering its importance, some classic model has been established like $ARCH$ and $GARCH$ almost accompany with the advent of Black-Scholes model, see [10][11].

Remark3. As pointed out in Section II, the solution of binomial model converges to solution of continuous model based on Black-Scholes model. The key point to hedge the option is replicate the pricing process, thus pricing the option becomes more essential. To illustrate this point of view more clearly, let's review our hedge scheme: in the whole hedging process, our goal is neutralize delta δ, which is directly related to option price. In this concrete model(binomial model), call price and put price can be easily worked out, which is not the same case for American put in continuous model(Black-Scholes model) since it does not have explicit solution formula see [1] chapter 5. Therefore, many algorithms have been developed to tackle this problem, which may be summarized to two major approaches. The first approach is a solution to the integral equation, where the option value is written as the expected value, under the risk neutral probability measure, of the option payoffs. Representative algorithms are Binomial method (the method we used in this project), see [3]; G-J(Geske and Johnson)method, see [6][8]; Accelerated Binomial method, see [7]. The second approach is to directly solve the

Black-Scholes (1973) partial differential equation see [1] chapter 5, subject to the boundary conditions imposed by the possibility of early exercise. Representative algorithms are Finite Difference [5] and Recursive method[9].

Remark4. In practice, delta (the changing rate of option price w.r.t. stock price) is one of the Greek letters taken into consideration, and other letters such as theta (the changing rate of option price w.r.t. time), gamma (the changing rate of delta w.r.t. time or the second derivatives of option price w.r.t. stock price), vega (the derivative of option price w.r.t. volatility), rho (the derivative of option price w.r.t. riskless interest rate). The following words are quoted from [2] "In ideal world, trader working for financial institutions would be able to rebalance their portfolios very frequently in order to maintain a zero delta, a zero vega, and so on. In practice, this is not impossible. When managing a large portfolio dependent on a single underlying asset, traders usually zero out delta at least once a day by trading the underlying asset. Unfortunately, a zero gamma and a zero vega are less easy to achieve because it is difficult to find options or other nonlinear derivatives that can be traded in the volume required at competitive prices (my understanding is that the real market price of option doesn't equal to theoretic one. As Cox, Ross and Rubinstein say 'To adjust a hedged position, never buy an overpriced option or sell an underpriced option.'). In most case, gamma and vega are monitored. When they get too large in a positive or negative direction, either corrective action is taken or trading is curtailed."

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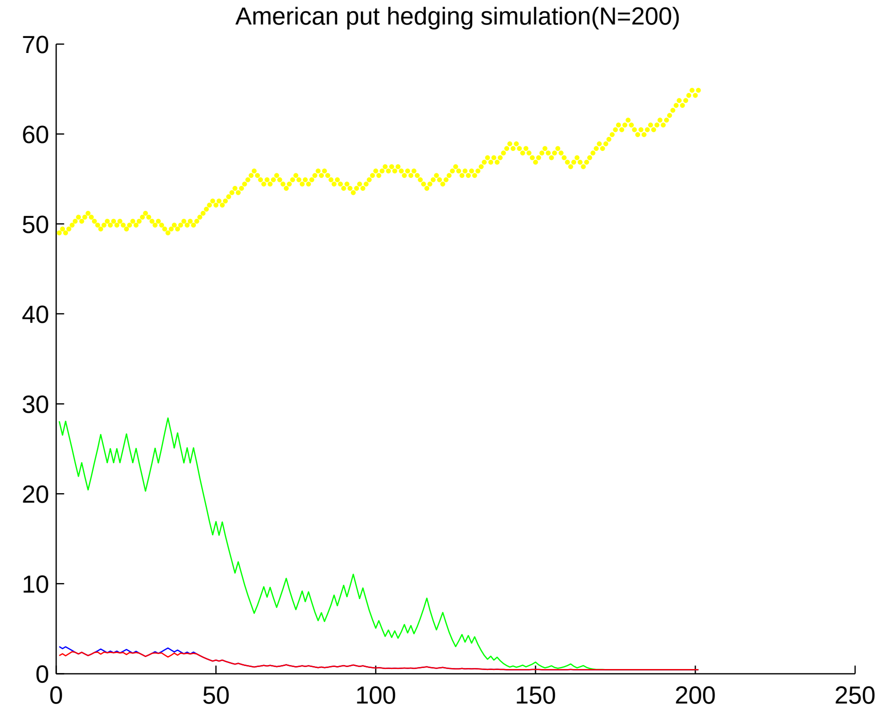
APPENDIX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50
40
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20
10
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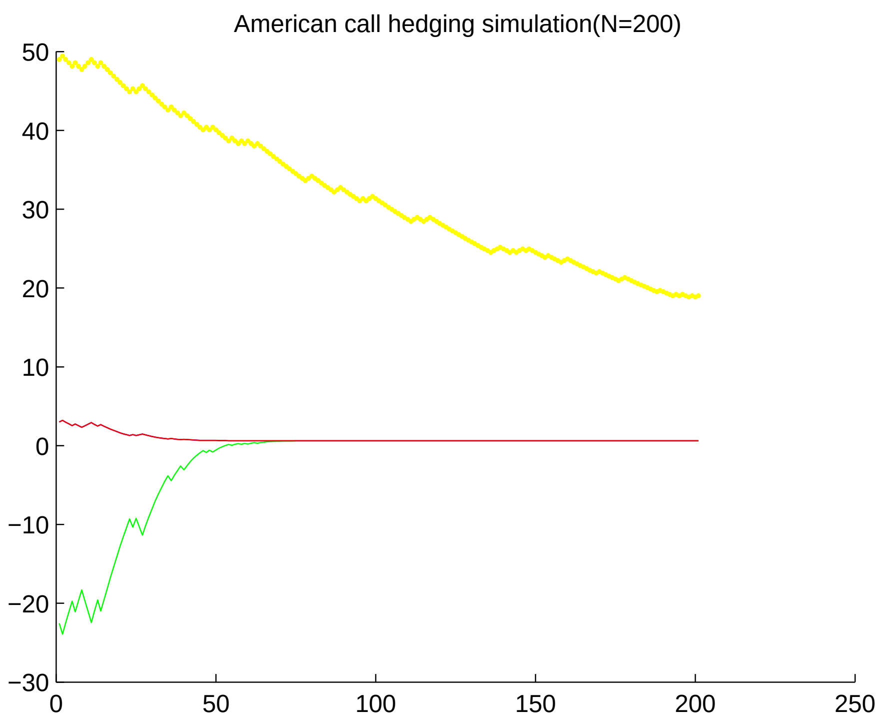
0
50
100
150
200
250


American call hedging simulation(N=200)
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−30
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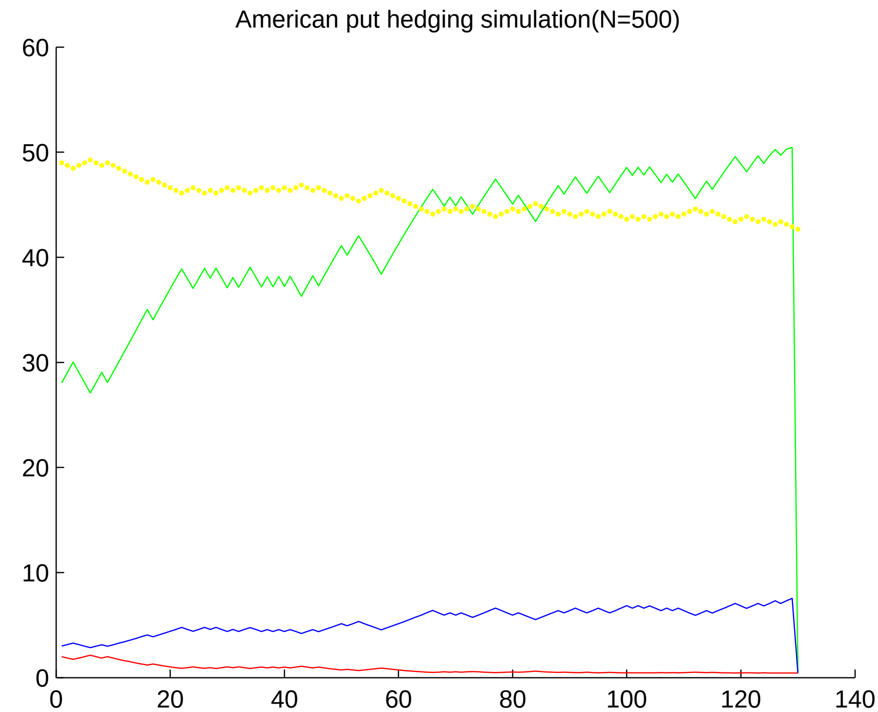
50
100
150
200
250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80
100
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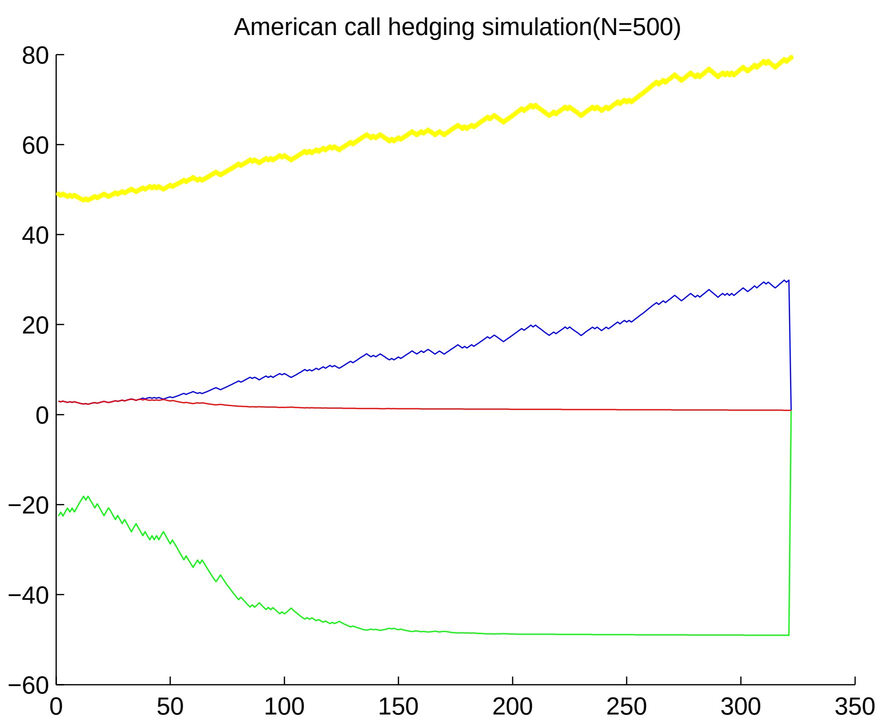
140


American call hedging simulation(N=500)
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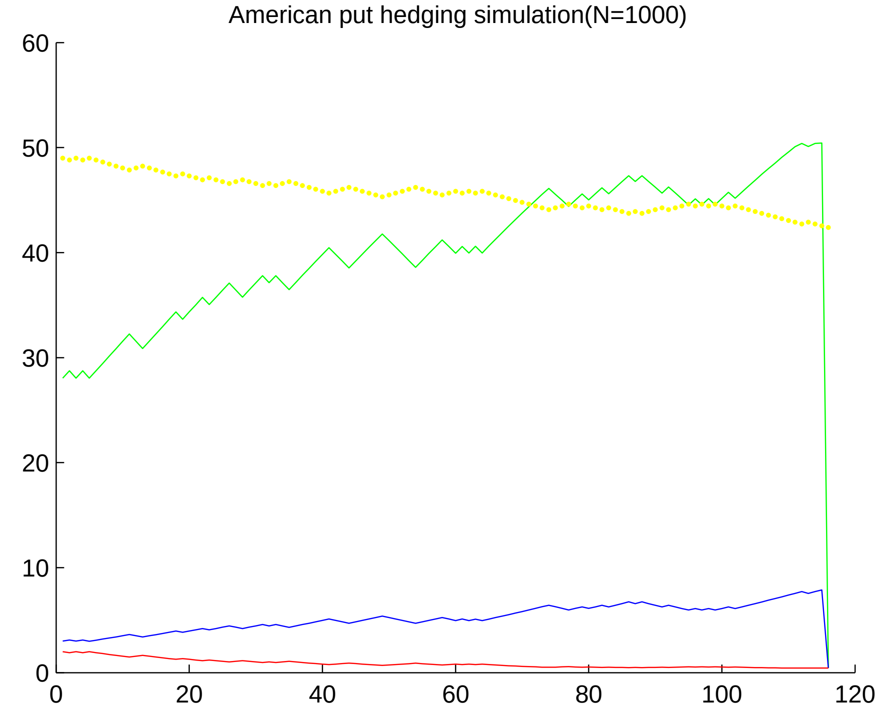
350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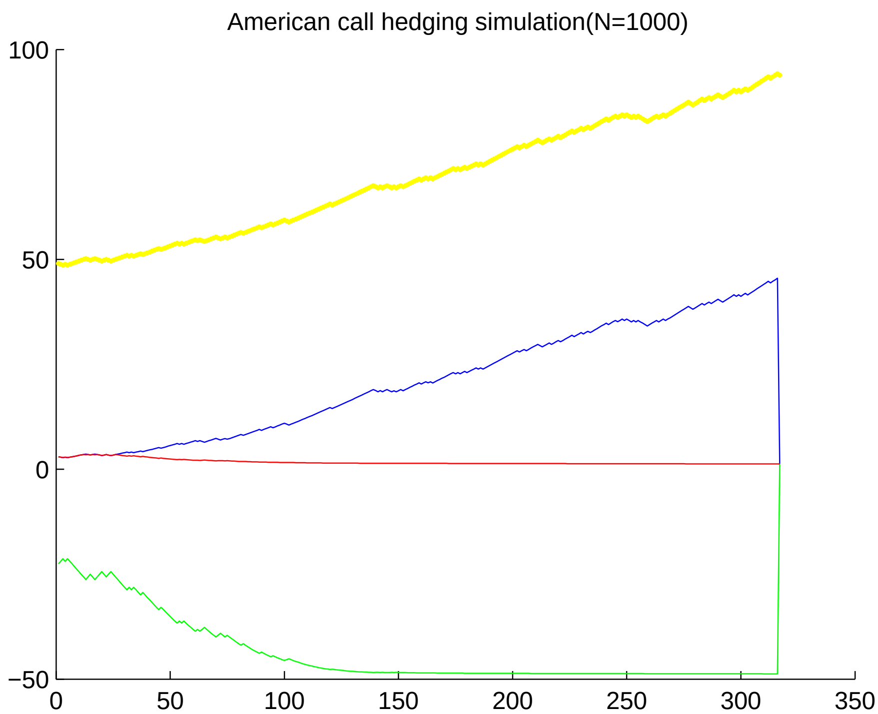
100
120


American call hedging simulation(N=1000)
100
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